

A functional central limit theorem for the partial sums of sorted i.i.d. random variables

Jean-François Marckert

David Renault

CNRS, LaBRI, Université de Bordeaux

351 cours de la Libération

33405 Talence cedex, France

email: name@labri.fr

Abstract

Let $(X_i, i \geq 1)$ be a sequence of i.i.d. random variables with values in $[0, 1]$, and f be a function such that $\mathbb{E}(f(X_1)^2) < +\infty$. We show a functional central limit theorem for the process $t \mapsto \sum_{i=1}^n f(X_i)1_{X_i \leq t}$.

Keywords : Empirical process, Donsker class

AMS classification : 62G30

1 Introduction

Let (X_1, X_2, \dots) be a sequence of i.i.d. random variables (r.v.) with values in $[0, 1]$, having distribution μ , distribution function F , and defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which the expectation operator is denoted \mathbb{E} . In this paper we are interested in proving a functional limit theorem for the sequence of processes $(Z_n, n \geq 1)$ defined by

$$Z_n(t) := \frac{1}{n} \sum_{i=1}^n f(X_i)1_{X_i \leq t}, \quad t \in [0, 1] \quad (1)$$

where $f : [0, 1] \rightarrow \mathbb{R}$ is a measurable function. Let $(\hat{X}_i, 1 \leq i \leq n)$ be the sequence $(X_i, 1 \leq i \leq n)$ sorted in increasing order, and for any $t \in [0, 1]$, denote by

$$N_n(t) = \#\{i : 1 \leq i \leq n, X_i \leq t\}$$

the number of X_i 's smaller than t . Clearly, for $t \in [0, 1]$,

$$Z_n(t) = \frac{1}{n} \sum_{k=1}^{N_n(t)} f(\hat{X}_k).$$

Hence, Z_n encodes the partial sums of functions of sorted i.i.d. r.v., as mentioned in the title of this paper. In order to state a central limit theorem for Z_n the existence of $\text{Var}(f(X_1)) < +\infty$ is clearly needed, but it is not sufficient to control the fluctuations of Z_n on all intervals. Standard

considerations about the binomial distribution implies that $N_n(t_2) - N_n(t_1)$ is quite concentrated around $n(F(t_2) - F(t_1))$ (for $t_1 < t_2$). Conditionally on $(N_n(t_1), N_n(t_2)) = (n_1, n_2)$,

$$Z_n(t_2) - Z_n(t_1) \stackrel{(d)}{=} \frac{1}{n} \sum_{k=1}^{n_2 - n_1} f(X_{(t_1, t_2]}(k)) \quad (2)$$

where $\stackrel{(d)}{=}$ means “equals in distribution”, and where $(X_{(t_1, t_2]}(k), 1 \leq k \leq n_2 - n_1)$ is a family of i.i.d. r.v., whose common distribution is that of X conditional on $X \in (t_1, t_2]$. Hence, to get a functional central limit theorem for Z_n , the variances of these distributions need to be controlled. The following hypothesis **Hyp** is designed for that purpose:

Hyp: there exists an increasing function $T : [0, 1] \rightarrow \mathbb{R}^+$ such that:

$$\begin{cases} x/T(x) \text{ is bounded,} \\ T(x) \ln(x) \xrightarrow{x \rightarrow 0} 0, \\ \forall I \text{ interval } \subset [0; 1], \quad \text{Var}(f(X) | X \in I) \leq \frac{T(\mu(I))}{\mu(I)} \end{cases}$$

where $\text{Var}(g(X) | X \in I)$ denotes the variance of $g(X)$ conditional on $X \in I$ (by convention, we set $\mathbb{E}(g(X) | X \in I) = 0$ when $\mathbb{P}(X \in I) = 0$).

When f is bounded by γ on $[0, 1]$, the function $T(x) = \gamma^2 x$ satisfies **Hyp** (see also the discussion below Theorem 1).

Consider the mean of Z_n

$$Z(t) := \mathbb{E}(Z_n(t)) = \mathbb{E}(f(X)1_{X \leq t}), \quad (3)$$

(this can be shown to be a càdlàg process when $\mathbb{E}(|f(X)|) < +\infty$) and

$$Y_n(t) = \sqrt{n} [Z_n(t) - Z(t)]. \quad (4)$$

The aim of this paper is to show the following result :

Theorem 1. *Let $(X_i, i \geq 0)$ be a sequence of i.i.d. r.v. taking their values in $[0, 1]$ and $f : [0, 1] \rightarrow \mathbb{R}$ a measurable function satisfying **Hyp**, then*

$$Y_n \xrightarrow[n]{(d)} Y$$

in $D[0, 1]$, the space of càdlàg functions on $[0, 1]$ equipped with the Skorokhod topology, where $(Y_t, t \in [0, 1])$ is a centered Gaussian process with variance function

$$\text{Var}(Y_s) = F(s)\text{Var}(f(X) | X \leq s) + F(s)(1 - F(s))\mathbb{E}(f(X) | X \leq s)^2 \quad (5)$$

and with covariance function, for $0 \leq s < t \leq 1$

$$\text{Cov}(Y_s, Y_t - Y_s) = -F(s)(F(t) - F(s))\mathbb{E}(f(X) | X \leq s)\mathbb{E}(f(X) | s < X \leq t). \quad (6)$$

We discuss a bit the conditions in the theorem. Assume that the X_i 's are i.i.d. uniform on $[0, 1]$, and that $f(x) = 1/x^\alpha$ for some $\alpha > 0$. The r.v. $f(X) = 1/X^\alpha$ possesses a variance iff $\alpha < 1/2$, and then it is in the domain of attraction of the normal distribution only in this case (Theorem 1 needs this hypothesis for the convergence of $Y_n(1)$). The largest $\text{Var}(f(X)|X \in (a, a + \varepsilon))$ is obtained for $a = 0$, in which case we get

$$\text{Var}(f(X) | X \in (0, \varepsilon]) = \frac{\varepsilon^{-2\alpha} \alpha^2}{(1 - 2\alpha)(1 - \alpha)^2},$$

and one can check that $\alpha < 1/2$ is also the condition for the existence of a function T satisfying Hyp. Hyp appears to be a minimal assumption in that sense.

The first result concerning the convergence of empirical processes is due to Donsker's Theorem [2]. It says that when f is constant equal to 1, then Y_n converges in $D[0, 1]$ to the standard Brownian bridge \mathbf{b} up to a time change. A kind of miracle arises then, since the same analysis works for all distributions μ by a simple time change. This is not the case here.

Apart from strong convergence theorems à la Komlós-Major-Tusnády [4], modern results about the convergence of empirical processes – see Shorack & Wellner [7] and van der Vaart & Wellner [8] – much rely on the concept of Donsker classes, which we discuss below.

Denote by $\mathbb{P}_n = \frac{1}{n} \sum_{k=1}^n \delta_{X_i}$ the empirical measure associated with the sample $(X_i, 1 \leq i \leq n)$. As a measure, \mathbb{P}_n operates on any set \mathcal{F} of measurable functions $\phi : [0, 1] \rightarrow \mathbb{R}$,

$$\mathbb{P}_n \phi = \int \phi(x) d\mathbb{P}_n(x) = \sum_{k=1}^n \phi(X_i)/n.$$

The empirical process is the signed measure $\mathbb{G}_n := \sqrt{n}(\mathbb{P}_n - \mu)$. By the standard central limit theorem, for a given function ϕ (such that $\mu\phi^2 < +\infty$), $\mathbb{G}_n \phi \xrightarrow[n]{(d)} \mathcal{N}(0, \mu(\phi - \mu\phi)^2)$, where $\mathcal{N}(m, \sigma^2)$ designates the normal distribution with mean m and variance σ^2 .

A *P-Donsker class* is a set of measurable functions \mathcal{F} such that $(\mathbb{G}_n \phi, \phi \in \mathcal{F})$ converges in distribution to $(\mathbb{G} \phi, \phi \in \mathcal{F})$, in the L_∞ topology (it is a central limit theorem for a process indexed by a set of functions). This means that :

- the convergence of the finite dimensional distributions holds : (meaning that for any k , any $\phi_1, \dots, \phi_k \in \mathcal{F}$, $(\mathbb{G}_n \phi_1, \dots, \mathbb{G}_n \phi_k) \xrightarrow[n]{(d)} N := (N_1, \dots, N_k)$ and N is a centered Gaussian vector with covariance matrix $\text{Cov}(N_i, N_j) = \mu[(\phi_i - \mu\phi_i)(\phi_j - \mu\phi_j)]$).
- the sequence $(\mathbb{G}_n \phi, \phi \in \mathcal{F})$ is tight in L_∞ .

The proof that a set forms a Donsker class is usually not that simple, and numerous criteria can be found in the literature. In our case, the set of functions \mathcal{F} is the following one :

$$\mathcal{F}_f = \{(x \mapsto \phi_t(x) = f(x)1_{x \leq t}), t \in [0, 1]\}.$$

We were unable to find such a criterion for this class, but notice that if such a result existed, it would imply Theorem 1 only for the topology L_∞ , a topology which is weaker than ours. Of course, Theorem 1 implies that \mathcal{F}_f forms a Donsker class.

Note. In fact classes \mathcal{F}_f for non decreasing f , or for functions f whose level sets are given by two intervals at most (such that $x \mapsto x^2$, $x \mapsto \cos(2\pi x)$, $x \mapsto \sin(2\pi x)$) are Donsker, since they are VC subgraph class (see Vapnik & Chervonenkis [9]).¹

If we consider the variables X_i 's in the formula (1), as random times, then $Z_n(t)$ corresponds (up to the normalisation) to the sum of $f(X_i)$ for all events X_i appearing before time t , where f is some cost function. The process Y_n appears to be the suitable tool to measure the fluctuations of Z_n .

We would like to mention [5], a work at the origin of the present paper, written by the same authors. In [5], the convergence of rescaled trajectories made with sorted increments (in \mathbb{C}) to a deterministic convex is shown. For this purpose a weaker version of Theorem 1 is established.

We provide a proof of our theorem in an old fashioned style. We prove the convergence of the finite dimensional distributions, and then establish the tightness in $D[0, 1]$; even if the proof is a bit technical, we think that several tricks make it interesting in its own right.

2 Proof of Theorem 1

The proof starts with that of the convergence of the finite dimensional distributions (FDD) convergence of Y_n : this is classical as we will see. Let $\theta_0 := 0 < \theta_1 < \theta_2 < \dots < \theta_K = 1$ for some $K \geq 1$ be fixed. In the sequel, for any function (random or not) L indexed by θ , $\Delta L(\theta_j)$ will stand for $L(\theta_j) - L(\theta_{j-1})$. For any $\ell \leq K$

$$\Delta Y_n(\theta_\ell) = \sqrt{n} [\Delta Z_n(N_n(\theta_j)) - \Delta Z(\theta_j)], \quad (7)$$

where by convention $Z_n(N_n(\theta_{-1})) = Z(\theta_{-1}) = 0$. The convergence of the FDD of Y_n follows the convergence in distribution of the increments $(\Delta Y_n(\theta_\ell), 0 \leq \ell \leq K)$. Notice that

$$\Delta Z(\theta_j) = \mathbb{E} (f(X) 1_{\theta_{j-1} < X \leq \theta_j}). \quad (8)$$

If for some j , θ_{j-1} and θ_j are chosen in such a way that $\Delta F(\theta_j) = 0$ then the j th increment in (7) is 0 almost surely (this is the case for the 0th increment if $\mu(\{0\}) = 0$). We now discuss the asymptotic behaviour of the other increments : let $J = \{j \in \{0, \dots, K\} : \Delta F(\theta_j) \neq 0\}$.

Let $(n_j, j \in J)$ be some fixed integers summing to n . Denote by $\mu_{\theta_{j-1}, \theta_j}$ the law of X conditioned by $\{\theta_{j-1} < X \leq \theta_j\}$. Conditional on $(N_n(\theta_j) = n_j, j \in J)$, the variables $\Delta Z_n(N_n(\theta_j))$, $j \in J$ are independent, and $\Delta Z_n(N_n(\theta_j))$ is a sum of $n_j - n_{j-1}$ i.i.d. copies of variables under $\mu_{\theta_{j-1}, \theta_j}$, denoted from now on $(X_{\theta_{j-1}, \theta_j}(k), k \geq 1)$.

¹We thank Emmanuel Rio for this information

Since $(\Delta N_n(\theta_j), j \in J) \sim \text{Multinomial}(n, (\Delta F(\theta_j), j \in J))$,

$$\left(\frac{\Delta N_n(\theta_j) - n\Delta F(\theta_j)}{\sqrt{n}}, j \in J \right) \xrightarrow[n]{(d)} (G_j, j \in J) \quad (9)$$

where $(G_j, j \in J)$ is a centered Gaussian vector with covariance function,

$$\text{cov}(G_k, G_\ell) = -\Delta F(\theta_k) \cdot \Delta F(\theta_\ell) + 1_{k=\ell} \Delta F(\theta_k),$$

formula valid for any $0 \leq k, \ell \leq K$. Putting together the previous considerations, we have

$$\Delta Y_n(\theta_j) = \sum_{m=1}^{\Delta N_n(\theta_j)} \frac{f(X_{\theta_{j-1}, \theta_j}(m)) - \mathbb{E}(f(X_{\theta_{j-1}, \theta_j}))}{\sqrt{n}} \quad (10)$$

$$+ \left(\frac{\Delta N_n(\theta_j) - n\Delta F(\theta_j)}{\sqrt{n}} \right) \mathbb{E}(f(X_{\theta_{j-1}, \theta_j})) \quad (11)$$

Using (9) and the central limit theorem, we then get that

$$(\Delta Y_n(\theta_j), 0 \leq j \leq K) \xrightarrow[n]{(d)} \left(\sqrt{\Delta F(\theta_j)} \tilde{G}_j + G_j \mathbb{E}(f(X_{\theta_{j-1}, \theta_j})), 0 \leq j \leq K \right) \quad (12)$$

where the family of r.v. $(G_j, j \leq K)$ and $(\tilde{G}_j, j \leq K)$ are independent, and the r.v. \tilde{G}_j are independent centered Gaussian r.v. with variance $\text{Var}(f(X_{\theta_{j-1}, \theta_j}))$ (this allows one to determine the variance and covariance (5) and (6)). Notice that here only the finiteness of $\text{Var}(f(X_{\theta_{j-1}, \theta_j}))$ and $\mathbb{E}(f(X_{\theta_{j-1}, \theta_j}))$ are used.

It remains to show the tightness of the sequence $(Y_n, n \geq 0)$ in $D[0, 1]$. A criterion for the tightness in $D[0, 1]$ can be found in Billingsley [1, Thm. 13.2]: a sequence of processes $(Y_n, n \geq 1)$ with values in $D[0, 1]$ is tight if, for any $\varepsilon \in (0, 1)$,

$$\lim_{\delta \rightarrow 0} \limsup_n \mathbb{P}(\omega'(Y_n, \delta) \geq \varepsilon) = 0$$

where $\omega'(f, \delta) = \inf_{(t_i)} \max_i \sup_{s, t \in [t_{i-1}, t_i]} |f(s) - f(t)|$, and the partitions (t_i) range over all partitions of the form $0 = t_0 < t_1 < \dots < t_n \leq 1$ with $\min\{t_i - t_{i-1}, 1 \leq i \leq n\} \geq \delta$.

We now compare our current model formed by a set $\{X_1, \dots, X_n\}$ of n i.i.d. copies of X denoted from now on by \mathbb{P}_n , with a Poisson point process P_n on $[0, 1]$ with intensity $n\mu$, denoted by \mathbb{P}_{P_n} . Conditionally on $\#P_n = k$, the k points $P_n := \{X'_1, \dots, X'_k\}$ are i.i.d. and have distribution μ , and then $\mathbb{P}_{P_n}(\cdot | \#P = n) = \mathbb{P}_n$. The Poisson point process is naturally equipped with a filtration $\sigma := \{\sigma_t = \sigma(\{P \cap [0, t]\}), t \in [0, 1]\}$.

We are here working under \mathbb{P}_{P_n} , and we let $N(\theta) = \#(P_n \cap [0, \theta])$; notice that under \mathbb{P}_n , N and N_n coincide.

Before starting, recall that if $N \sim \text{Poisson}(b)$, for any positive λ ,

$$\mathbb{P}(N \geq x) = \mathbb{P}(e^{\lambda N} \geq e^{\lambda x}) \leq \mathbb{E}(e^{\lambda N - \lambda x}) = e^{-b + be^{\lambda} - \lambda x} \quad (13)$$

$$\mathbb{P}(N \leq x) = \mathbb{P}(e^{-\lambda N} \geq e^{-\lambda x}) \leq \mathbb{E}(e^{-\lambda N + \lambda x}) = e^{-b + be^{-\lambda} + \lambda x}. \quad (14)$$

We explain now why the tightness of $(Y_n, n \geq 1)$ under \mathbb{P}_{P_n} implies the same result under \mathbb{P}_n . Let $m = \inf\{x \in [0, 1], F(x) \geq 1/2\}$ be the median of μ .

Lemma 2.1. *There exists a constant γ (which depends on μ), such that for any σ_m -measurable event A ,*

$$\mathbb{P}_n(A) = \mathbb{P}_{P_n}(A \mid \#P = n) \leq \gamma \mathbb{P}_{P_n}(A). \quad (15)$$

Proof of the Lemma We have

$$\begin{aligned} \mathbb{P}_{P_n}(A \mid \#P = n) &= \sum_k \frac{\mathbb{P}_{P_n}(A, \#(P \cap [0, m]) = k) \mathbb{P}(\#P \cap [m, 1] = n - k)}{\mathbb{P}(\#P = n)} \\ &\leq \sum_k \mathbb{P}_{P_n}(A, \#(P \cap [0, m]) = k) \sup_{k'} \frac{\mathbb{P}(\#P \cap [m, 1] = n - k')}{\mathbb{P}(\#P = n)} \\ &\leq \gamma \mathbb{P}_{P_n}(A) \end{aligned}$$

where $\gamma = \sup_{n \geq 1} \sup_{k'} \frac{\mathbb{P}(\#P \cap [m, 1] = n - k')}{\mathbb{P}(\#P = n)}$, which is indeed finite since $\mathbb{P}(\#P = n) \sim (2\pi n)^{-1/2}$, and since $\#P \cap [m, 1] \sim \text{Poisson}(n/2)$, and then the probability that its value is k is bounded above by some d/\sqrt{n} according to Petrov [6, Thm. 7 p. 48]. \square

Thanks to Lemma 2.1, if the sequence of restrictions $(Y_n|_{[0, m]}, n \geq 1)$ of Y_n on $[0, m]$ is tight on $D[0, m]$ under \mathbb{P}_{P_n} then so it is under \mathbb{P}_n (the same proof works on $D[m, 1]$ by a time reversal argument). To end the proof, we show that $(Y_n|_{[0, m]}, n \geq 1)$ is indeed tight under \mathbb{P}_{P_n} .

Take then some (small) $\eta \in (0, 1)$, $\varepsilon > 0$; we will show that one can find a finite partition $(t_i, i \in I)$ of $[0, m]$ and a $\delta \in (0, m)$ such that

$$\limsup_n \mathbb{P}_n(\omega'(Y_n, \delta) \geq \varepsilon) \leq \eta, \quad (16)$$

which is sufficient for our purpose.

We decompose the process Y_n as suggested by (10) and (11),

$$Y_n(\theta) = Y'_n(\theta) + Y''_n(\theta) \quad (17)$$

where

$$Y'_n(\theta) = \sum_{m=1}^{N_n(\theta)} \frac{f(X_{[0, \theta]}(m)) - \mathbb{E}(f(X_{[0, \theta]}))}{\sqrt{n}} \quad (18)$$

$$Y''_n(\theta) = \left(\frac{N_n(\theta) - nF(\theta)}{\sqrt{n}} \right) \frac{Z_\theta}{F(\theta)}. \quad (19)$$

(If $F(\theta)$ then set $Y''_n(\theta) = 0$ instead of (19)).

The tightness of each of the sequences $(Y'_n, n \geq 1)$ and $(Y''_n, n \geq 1)$ in $D[0, 1]$ suffices to deduce that of $(Y_n, n \geq 1)$. We then proceed separately.

Tightness of $(Y'_n, n \geq 1)$

To control the jumps of Y'_n , we will need to localise the large atoms of μ . Let $A = \{x \in [0, m], \mu(\{x\}) > 0\}$ be the set of positions of the atoms of μ in $[0, m]$, and let $A^{\geq a} := \{x \in A : \mu(\{x\}) \geq a\}$. Clearly $\#A^{\geq a} \leq 1/a$ and $[0, m] \setminus A^{\geq a}$ forms a finite union of open connected intervals $(O_x, x \in G)$, with extremities $(t'_i, i \in I)$. The intervals $(O_x, x \in G)$ can be further cut as follows:

- do nothing to those such that $\mu(O_x) < 2a$,
- those such that $\mu(O_x) > 2a$ are further split. Since they contain no atom with mass $> a$, they can be split into smaller intervals having all their weights in $[a, 2a]$ except for at most one (in each interval O_x which may have a weight smaller than a).

Once all these splittings have been done, a list of at most $3/a$ intervals are obtained (in fact less than that), all of them having a weight smaller than $2a$. Name $G_a = (O_x, x \in I_a)$ the collection of obtained open intervals, indexed by some set I_a , and by $(t_i^a, i \geq 0)$ the partitions obtained. Take O one of these intervals. One has $\#(P_n \cap O)$ is Poisson with parameter $n\mu(O) \leq na$. Consider again (10), (11) and Hyp. Set, for any $L \geq 1$,

$$S_L^{(n)} := \sum_{\ell=1}^L \frac{f(X_O(\ell)) - \mathbb{E}(f(X_O))}{\sqrt{n}}.$$

Let

$$\omega(Y'_n, O) = \sup\{|Y'_n(s) - Y'_n(t)|, s, t \in O\}$$

be the modulus of continuity of Y'_n on O . We have, for any $\alpha \in (0, 1/2)$,

$$\begin{aligned} \mathbb{P}(\omega(Y'_n, O) \geq x) &\leq \mathbb{P}\left(|\#(P_n \cap O) - n\mu(O)| \geq n^{1/2+\alpha}\right) \\ &\quad + \sup_{L \in \Gamma_n} \mathbb{P}\left(\sup\left\{|S_i^{(n)} - S_j^{(n)}|, i, j \leq L\right\} \geq x\right) \end{aligned} \quad (20)$$

where

$$\Gamma_n = \left[n\mu(O) - n^{1/2+\alpha}, n\mu(O) + n^{1/2+\alpha}\right].$$

Using (13) and (14), one sees that

$$\mathbb{P}\left(|P(n\mu(O)) - n\mu(O)| \geq n^{\alpha+1/2}\right) \leq ce^{-c'n^\alpha}$$

for some $c > 0, c' > 0$ and n large enough (for this take $x = n\mu(O) + n^{1/2+\alpha}$, $\lambda = 1/\sqrt{n}$ in (13) and, $x = n\mu(O) - n^{1/2+\alpha}$, $\lambda = 1/\sqrt{n}$ in (14)).

Let us take care of the second term in (20). Clearly,

$$\sup\left\{|S_i^{(n)} - S_j^{(n)}|, i, j \leq L\right\} = \max_{i \leq L} S_i^{(n)} - \min_{j \leq L} S_j^{(n)}.$$

According to Petrov [6, Thm.12 p50],

$$\mathbb{P}\left(\max_{i \leq L} S_i^{(n)} \geq x\right) \leq 2\mathbb{P}\left(S_L^{(n)} \geq x - \sqrt{\frac{2L\text{Var}(f(X_O))}{n}}\right),$$

and then

$$\mathbb{P}\left(\max_{i \leq L} S_i^{(n)} \geq x\right) \leq 2\mathbb{P}\left(S_L^{(n)} \geq x - C_n(O)\right),$$

for $C_n(O) = \sqrt{\frac{2LT(\mu(O))}{n\mu(O)}}$, and a similar inequality holds for $\min_{i \leq L} S_i^{(n)}$. Since

$$\begin{aligned} \mathbb{P}\left(\max_{i \leq L} S_i^{(n)} - \min_{j \leq L} S_j^{(n)} \geq x\right) &\leq \mathbb{P}\left(\max_{i \leq L} S_i^{(n)} \geq x/2\right) + \mathbb{P}\left(-\min_{j \leq L} S_j^{(n)} \geq x/2\right) \\ &\leq 2\mathbb{P}\left(S_L^{(n)} \geq \frac{x}{2} - C_n(O)\right) + 2\mathbb{P}\left(S_L^{(n)} \leq -\frac{x}{2} + C_n(O)\right). \end{aligned}$$

To get some bounds, we use the central limit theorem for $S_L^{(n)}$, and take $x = \varepsilon$, $a > 0$ such that $T(a) = \varepsilon^2 \delta^2$ for some small $\delta > 0$ (recall that T is increasing and therefore invertible), and any sequence L_n such that $L_n/n \rightarrow \mu(O)$ (any sequence $L = L_n$ such that $L_n \in \Gamma_n$ satisfies this, and then we can control the supremum with this method). We have

$$\mathbb{P}\left(S_L^{(n)} \geq \frac{\varepsilon}{2} - C_n(O)\right) = \mathbb{P}\left(\frac{S_L^{(n)}}{\sqrt{\mu(O)\text{Var}(f(X_O))}} \geq \frac{\varepsilon/2 - C_n(O)}{\sqrt{\mu(O)\text{Var}(f(X_O))}}\right).$$

For n large enough,

$$C_n(O) \leq \sqrt{4T(\mu(O))} \leq 2\varepsilon\delta$$

and therefore

$$\limsup_n \mathbb{P}\left(S_L^{(n)} \geq \frac{\varepsilon}{2} - C_n(O)\right) \leq \Phi\left(\frac{\varepsilon/2 - 2\varepsilon\delta}{\sqrt{\mu(O)\text{Var}(f(X_O))}}\right)$$

where Φ is the tail function of the standard Gaussian distribution.

Finally, if δ is chosen sufficiently small ($2\delta < 1/2$), since $\mu(O)\text{Var}(f(X_O)) \leq T(\mu(O)) \leq T(a) = \varepsilon^2 \delta^2$, then on each interval $O \in G_a$,

$$\mathbb{P}\left(\sup\left\{|S_i^{(n)} - S_j^{(n)}|, i, j \leq L\right\} \geq \varepsilon\right) \leq 4\Phi\left(\frac{1/2 - 2\delta}{\delta}\right)$$

and this independently of the choice of the interval O in G_a , for n large enough.

The control of the intervals all together can be achieved using the union bound : since they are at most $3/T^{-1}(\varepsilon^2 \delta^2)$ such intervals, by the union bound

$$\mathbb{P}_{P_n}\left(\sup_{O \in G_a} \omega(Y'_n, O) \geq \varepsilon\right) \leq \frac{3}{T^{-1}(\varepsilon^2 \delta^2)} \left(4\Phi\left(\frac{1/2 - 2\delta}{\delta}\right) + ce^{-c'n^\alpha}\right).$$

Since $\Phi(x) \underset{x \rightarrow +\infty}{\sim} \exp(-x^2/2)/(\sqrt{2\pi}x)$, and $T(x) \ln(x) \underset{x \rightarrow 0}{\rightarrow} 0$, which implies that for any $\varepsilon > 0$, and $\gamma > 0$ there exists a δ sufficiently small such that

$$T(e^{-\gamma/\delta^2}) < \varepsilon^2 \delta^2 \quad \text{or equivalently} \quad \frac{1}{T^{-1}(\varepsilon^2 \delta^2)} < e^{\gamma/\delta^2}$$

and as a result the probability can be taken as small as wanted. \square

Tightness of $(Y_n'', n \geq 1)$

Recall (19). We work here under \mathbb{P}_n and we only consider the interval $I = \{\theta : F(\theta) > 0\}$ since $Y_n''(\theta)$ equals 0 on its complement. Since on I , $\theta \mapsto \frac{Z_\theta}{F(\theta)}$ is càdlàg (and does not depend on n), it suffices to see why $\left(\frac{N_n(\theta) - nF(\theta)}{\sqrt{n}}, n \geq 0\right)$ is tight in $D[0, 1]$, but this is clear since this is a consequence of the convergence of the standard empirical process (Donsker [2]). \square

References

- [1] P. Billingsley, (1999) *Convergence of probability measures*, Wiley Series in Probab. and Stat, John Wiley & Sons Inc.
- [2] M.D. Donsker, (1952) *Justification and extension of Doob's heuristic approach to the Kolmogorov-Smirnov theorems*, Annals of Mathematical Statistics., 23:277–281, 1952
- [3] O. Kallenberg (1997), *Foundations of Modern Probability*. Probability and Its Applications. Springer, New York, NY.
- [4] J. Komlós, P. Major, G. Tusnády, (1975) *An approximation of partial sums of independent RV's and the sample DF*, I. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 32, 111–131.
- [5] J.F. Marckert, D. Renault (2012) *Compact convexes of the plane and probability theory*, preprint, arXiv:1208.5408.
- [6] V.V. Petrov (1975) *Sums of independent random variables*, Springer-Verlag, New York.
- [7] G.R. Shorack, J.A. Wellner, (1986) *Empirical processes with applications to statistics*, Wiley Series in Probab. and Math. Stat.: Probab. and Math. Stat. John Wiley & Sons, Inc., New York.
- [8] A.W. van der Vaart, J.A. Wellner, Jon A. (1996) *Weak convergence and empirical processes. With applications to statistics*. Springer Series in Stat. Springer-Verlag, New York.
- [9] V.N. Vapnik, A.YA. Chervonenkis (1971) *On the unifor convergence of relative frequencies of events to their probabilities*, Theory of probab. and appl., Vol. XVI, 2, 264–280.